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DYNAMIC RESPONSE OF ELLIPTICAL FOOTINGS

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Abstract—Dynamic response of an elliptical footing in frictionless contact with a homogeneous elastic half-space is considered. Both vertical and horizontal vibrations are treated. In the case of the vertical vibration, the mixed boundary value problem gives rise to a set of dual integral equations. For the horizontal vibration, we have a system of dual integral equations. The dual integral equations which are two dimensional in nature are reduced to two-dimensional Fredholm integral equations of the first kind. They are then recast in a suitable form after separating out the static solution. Successive low-frequency terms are then obtained by utilising the static solution. The series solutions up to ω^2, ω being the frequency, are obtained, and analytical results for the dynamic compliances are obtained. In the limiting case of a circular footing, our results are in agreement with those of previous authors.

1. INTRODUCTION

Analytical results on the dynamical response of footings and soil structure interaction are usually available for circular footings and infinite strip loadings. Such results have important bearings on the stability of buildings and vibration of dams and similar large structures subjected to earthquakes. Awojobi and Grootenhuis[1], Robertson[2], Gladwell[3] and others have considered the problem of circular footings in detail. The dual integral equations that arise in such mixed value problems are usually solved by Noble's technique[4], and power series solutions, valid in the low-frequency range, are usually obtained. Awojobi and Grootenhuis used a series expansion procedure to solve the same dual integral equation. Shah[5], and later Luco and Westmann[6], solved numerically the Fredholm integral equation and obtained the dynamical compliances for a wide range of physically important frequencies. A list of available references can be had in [6].

To improve the dynamic models of buildings and other structures, it will be fruitful to have analytical results for foundations other than circular ones. In this paper we develop a method for elliptical footings. Both vertical and horizontal vibrations are treated. In the former case, the vertical displacement is prescribed while the disc is in frictionless contact with the elastic half-space. In this case, we have a two-dimensional dual integral equation. In the latter case when the horizontal displacement is prescribed, we have a system of dual integral equations. Noble's technique, which could be easily adapted to solve dual integral equations that arise in the circular case, is no longer useful in our case because the dual integral equations are two dimensional in nature. We first reduce the dual integral equation to a Fredholm integral equation of the first kind which is then rearranged in a suitable form so that successive low-frequency terms can be generated once the static solution is known. We obtain first two terms of the power series up to k_2^2. Analytical expressions for the dynamical compliances of the elliptic disc experiencing vertical and transverse horizontal vibrations are obtained. We note that in the case of vertical vibration, Stallybrass and Scherer[7] used a variational approach to obtain the analytical expression for the reciprocal of the total load under the disc. Our results are in agreement with theirs up to the order of terms retained. For the horizontal vibration our results are believed to be new.

2. BASIC EQUATIONS

Let the footing be modeled as a rigid elliptic disc resting on a homogeneous elastic half-space. With reference to a Cartesian coordinate system (x, y, z), with origin at the centre of the elliptic disc, let the region bounded by the elliptic disc, which is in frictionless contact with the elastic half-space, $z \geq 0$, be given by

$$S: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, \qquad z = 0, \tag{1}$$

where a, b $(<a)$ are the semiaxes of the ellipse.

The displacement $\mathbf{u}(u_x, u_y, u_z)$ is given in terms of the potentials ϕ, ψ and χ as

$$\mathbf{u} = \nabla\phi + \nabla \times \nabla \times (\mathbf{e}_z\psi) + \nabla \times (\mathbf{e}_z\chi), \tag{2}$$

where

$$\begin{aligned} \phi &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_1(\xi, \eta)\, e^{-i(\xi x + \eta y) - \nu_1 z}\, d\zeta\, d\eta, \\ \psi &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_1(\xi, \eta)\, e^{-i(\xi x + \eta y) - \nu_2 z}\, d\xi\, d\eta, \\ \chi &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_1(\xi, \eta)\, e^{-i(\xi x + \eta y) - \nu_2 z}\, d\xi\, d\eta, \end{aligned} \tag{3}$$

where $\mathbf{e}_z$ is unit vector along the z-direction and

$$\nu_1 = (k^2 - k_1^2)^{1/2}, \qquad \nu_2 = (k^2 - k_2^2)^{1/2} \tag{4}$$

$k = (\xi^2 + \eta^2)^{1/2}$, $k_1 = \omega/\alpha$ and $k_2 = \omega/\beta$; α and β are the P- and S-wave velocities of the medium; the harmonic time dependence, $e^{i\omega t}$, will henceforth be suppressed.

For the solution to be outgoing at infinity, we choose the branch cuts, outwards to infinity, along the real axis with

$$\nu_j = \begin{cases} (k^2 - k_j^2)^{1/2}, & k \geq k_j, \\ i(k_j^2 - k^2)^{1/2}, & -k_j < k < k_j, \\ -(k^2 - k_j^2)^{1/2}, & k \leq -k_j \end{cases} \tag{5}$$

for $j = 1, 2$. The displacement and stress components are, from (2),

$$\begin{aligned} u_x &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [-i\xi\{A_1(\xi,\eta)e^{-\nu_1 z} - \nu_2 B_1(\xi,\eta)e^{-\nu_2 z}\} \\ &\quad - i\xi C_1(\xi,\eta)e^{-\nu_2 z}]\, e^{-i(\xi x + \eta y)}\, d\xi\, d\eta, \\ u_z &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [-\nu_1 A_1(\xi,\eta)e^{-\nu_1 z} + (\xi^2 + \eta^2)B_1(\xi,\eta)e^{-\nu_2 z}] \\ &\quad \times e^{-(\xi x + \eta y)}\, d\xi\, d\eta, \\ \tau_{zx} &= \mu\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [-i\xi\{-2\nu_1 A_1(\xi,\eta)e^{-\nu_1 z} + (2\xi^2 + 2\eta^2 - k_2^2)e^{-\nu_2 z} \\ &\quad \times B_1(\xi,\eta)\} + i\eta\nu_2 C_1(\xi,\eta)e^{-\nu_2 z}]\, e^{-i(\xi x + \eta y)}\, d\xi\, d\eta, \\ \tau_{zz} &= \mu\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(2\xi^2 + 2\eta^2 - k_2^2)A_1(\xi,\eta)e^{-\nu_1 z} - 2\nu_2(\xi^2 + \eta^2) \\ &\quad \times B_1(\xi,\eta)e^{-\nu_2 z}]\, e^{-i(\xi x + \eta y)}\, d\xi\, d\eta. \end{aligned} \tag{6}$$

u_y and τ_{zy} can be obtained from the expressions for u_x and τ_{zx}, respectively, upon changing ξ by η and η by $(-\xi)$.

3. VERTICAL VIBRATION OF THE DISC

We consider first the problem in which the disc performs vertical vibration of constant amplitude w_0. Boundary conditions in this case are

$$\tau_{zz} = 0, \qquad (x, y) \notin S, \tag{7a}$$

$$\tau_{zx} = 0 = \tau_{zy} \qquad \text{for all } x, y, \tag{7b}$$

$$u_z = w_0, \qquad (x, y) \in S. \tag{7c}$$

Condition (7b) gives

$$C_1(\xi, \eta) = 0, \qquad 2\nu_1 A_1(\xi, \eta) - (2\xi^2 + 2\eta^2 - k_2^2)B_1(\xi, \eta) = 0. \tag{8}$$

Using (8), (7a) and (7c) and defining a new unknown $A(\xi, \eta)$ as

$$A(\xi, \eta) = \frac{F(\xi, \eta)}{\nu_1} B_1(\xi, \eta), \tag{9}$$

where

$$F(\xi, \eta) = (\xi^2 + \eta^2 - k_2^2/2)^2 - (\xi^2 + \eta^2)\nu_1\nu_2, \tag{10}$$

we obtain the following dual integral equations:

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi, \eta) e^{-i(\xi x + \eta y)} \, d\xi \, d\eta = 0, \qquad (x, y) \notin S, \\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{A(\xi, \eta) k_2^2 \nu_1}{2F(\xi, \eta)} e^{-i(\xi x + \eta y)} \, d\xi \, d\eta = w_0, \qquad (x, y) \in S. \end{aligned} \tag{11}$$

4. FORMAL SOLUTION

If the frequency tends to zero, one obtains the dual integral equations for the static loading. We represent the static solution by $A_0(\xi, \eta)$ as $\omega \to 0$, where

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_0(\xi, \eta) e^{-i(\xi x + \eta y)} \, d\xi \, d\eta = g_0(x, y),$$

where

$$g_0(x, y) = 0, \qquad (x, y) \notin S. \tag{12}$$

Then

$$A_0(\xi, \eta) = \frac{1}{2\pi}\iint_S g_0(x', y') e^{i(\xi x' + \eta y')} \, dx' \, dy'. \tag{13}$$

Setting (13) in the limiting form of (11), as $\omega \to 0$, one obtains the following Fredholm integral equation for the determination of $g_0(x, y)$:

$$\iint_S \frac{g_0(x', y')}{R} \, dS = -\frac{w_0}{2(1 - \nu)}, \qquad (x, y) \in S, \tag{14}$$

where we have used the relation

$$\frac{1}{R} = \frac{1}{[(x - x')^2 + (y - y')^2]^{1/2}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i\xi(x-x') - i\eta(y-y')}}{(\xi^2 + \eta^2)^{1/2}} d\xi\, d\eta. \tag{15}$$

ν is Poisson's ratio.

When S is an ellipse, the form of (14) is similar to that encountered in the acoustic diffraction by a soft elliptic disc[8]. Following [8] we set

$$g_0(x, y) = \frac{c_0 H(1 - x^2/a^2 - y^2/b^2)}{(1 - x^2/a^2 - y^2/b^2)^{1/2}}, \tag{16}$$

where $H(z)$ is the Heaviside unit function. Then using the value of the integral listed in the Appendix, we get

$$c_0 = -w_0/4\pi b(1 - \nu)K, \tag{17}$$

where K is the elliptic integral of the first kind with argument $k_0 = (1 - b^2/a^2)^{1/2}$.

In order to get a solution of (11), we set

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi, \eta)e^{-i(\xi x + \eta y)} d\xi\, d\eta = g(x, y), \tag{18}$$

where $g(x, y) = 0$, $(x, y) \in S$.

Then the first equation in (11) is automatically satisfied. Substituting (18) in the second equation of (11), separating out the static solution, and rearranging, we get

$$-2(1 - \nu)\iint_S \frac{g(x', y')}{R} dS + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left\{\frac{k_2^2 \nu_1}{2F(\xi, \eta)} + \frac{2(1 - \nu)}{(\xi^2 + \eta^2)^{1/2}}\right\} \times \iint_S g(x', y')e^{-i\xi(x-x') - i\eta(y-y')} dx'\, dy'\right] d\xi\, d\eta = w_0 \quad (x, y) \in S. \tag{19}$$

Integral equation (19) will now be solved for the low-frequency case. To this end we assume, for $k_2 a \ll 1$, that $g(x, y)$ has the low-frequency expansion

$$g(x, y) = g_0(x, y) + \omega g_1(x, y) + \omega^2 g_2(x, y) + \cdots, \tag{20}$$

where $g_0(x, y)$ is given by (16).

We note that, by virtue of choice, the integrand in braces in the second integral in (19) vanishes as $\omega \to 0$. Thus the low-frequency expansion of the integrand starts with a first order in ω.

Hence, substituting (20) in (19) and equating like powers of ω, the equation determining $g_1(x, y)$ is then obtained as

$$\iint_S \frac{g_1(x', y')}{R} dS = J_1. \tag{21}$$

ωJ_1 is the first term in the low-frequency expansion of I_1.

$$I_1 = \frac{1}{4\pi(1-\nu)}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{k_2^2\nu_1}{F(\xi,\eta)}\,e^{-i(\xi x+\eta y)}\right.\right.$$
$$\left.\left.\times\iint_S g_0(x',y')e^{i(\xi x'+\eta y')}\,dx'\,dy'\right\}d\xi\,d\eta\right]$$
$$+\frac{1}{2\pi}\iint_S \frac{g_0(x',y')}{(\xi^2+\eta^2)^{1/2}}\,e^{-i\xi(x-x')-i\eta(y-y')}\,dx'\,dy'.$$

On using (15) and (14) the alternative form of I_1 is

$$I_1 = \frac{1}{4\pi(1-\nu)}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{k_2^2\nu_1}{F(\xi,\eta)}\,e^{-i(\xi x+\eta y)}\right.\right.$$
$$\left.\left.\times\iint_S g_0(x',y')e^{i(\xi x'+\eta y')}\,dx'\,dy'\right\}d\xi\,d\eta\right]-\frac{w_0}{2(1-\nu)}. \quad (22)$$

Similarly, the equation determining $g_2(x, y)$ is given by

$$\iint_S \frac{g_2(x',y')}{R}\,dS = J_2, \quad (23)$$

where $\omega^2 J_2$ is the first term in the low-frequency expansion of I_2, which is given by

$$I_2 = \frac{1}{4\pi(1-\nu)}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{k_2^2\nu_1}{F(\xi,\eta)}\right.$$
$$\left.\times\left\{\iint_S [g_0(x',y')+\omega g_1(x',y')]e^{-i\xi(x-x')-i\eta(y-y')}\,dx'\,dy'\right\}d\xi\,d\eta\right]$$
$$-\frac{w_0}{2(1-\nu)}-\omega J_1. \quad (24)$$

In a similar fashion any $g_n(x, y)$ is related to lower orders $g_{n-1}(x, y)$, etc. through a similar integral equation.

We now obtain the first-order expansion of the integral I_1. Using

$$\xi = k\cos\chi, \qquad \eta = k\sin\chi,$$
$$x = r\cos\phi, \qquad y = r\sin\phi,$$

the values of $g_0(x, y)$ from (16), and some standard formulae for Bessel functions give, after simplification.

$$I_1 = I_{11} + I_{12} - w_0/2(1-\nu). \quad (25)$$

In (25) we have

$$I_{11} = \frac{c_0 ab}{2(1-\nu)}\int_0^{2\pi}\int_0^{\infty}\frac{k_2^2\nu_1}{2F(k)}J_0(kr)\,\frac{\sin[k(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}]}{(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}\,dk\,d\chi,$$
$$I_{12} = -\frac{c_0 ab}{2(1-\nu)}\sum_{n=1}^{\infty}\int_0^{2\pi}\int_0^{\infty}\frac{k_2^2\nu_1}{2F(k)}J_{2n}(kr) \quad (26)$$
$$\times\frac{\sin[k(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}]}{(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}\cos 2n\chi\,dk\,d\chi\,(\cos 2n\phi),$$

where

$$F(k) = (k^2 - k_2^2/2)^2 - k^2(k^2 - k_1^2)^{1/2}(k^2 - k_2^2)^{1/2}. \tag{27}$$

Following [1] and [3] and considering an appropriate integral in the upper half-plane, another form of I_{11} is

$$I_{11} = -\frac{c_0 ab}{2(1-\nu)}\int_0^{2\pi}\left\{\int_0^{k_1}\frac{k_2^2(k_1^2-k^2)^{1/2}\,e^{-ik(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}}{2[(k^2-k_2^2/2)^2+(k_1^2-k^2)(k_2^2-k^2)k^2]}J_0(kr)\,dk\right.$$
$$+\int_{k_1}^{k_2}\frac{(k_2^2-k^2)^{1/2}k_2^2k^2(k^2-k_1^2)J_0(kr)}{2[(k^2-k_2^2/2)^4+k^4(k_1^2-k^2)(k_2^2-k^2)]}e^{-ik(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}\,dk$$
$$\left.-\frac{\pi(S_R^2-k_1^2)^{1/2}}{2F'(S_R)}k_2^2\,e^{-iS_R(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}J_0(S_R r)\right\}\frac{d\chi}{(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}.$$

$$I_{12} = -\sum\{\ \}, \tag{28}$$

where $\{\ \}$ is the term on the r.h.s. of I_{11}, except that $J_0(kr)$ is replaced by $J_{2n}(kr) \times \cos 2n\chi \cos 2n\phi$. In (28), s_R is the real root of $F(k) = 0$. Expanding (28) in powers of ω we get

$$I_1 = -c_0F_0 2bK + ik_2F_1abc_0\pi - \frac{w_0}{2(1-\nu)} + o(k_2^2)$$

where, for $n = 0, 1, 2, \ldots,$

$$F_n = \frac{1}{2(1-\nu)}\left[\int_0^1 \mathcal{K}(t)t^{n-1}\,dt - \frac{\pi s^2(s^2-\gamma^2)^{1/2}}{F'(s)}\right],$$

$$\mathcal{K}(t) = \begin{cases} \dfrac{t(\gamma^2-t^2)^{1/2}}{(t^2-\frac{1}{2})^2+t^2(\gamma^2-t^2)^{1/2}(1-t^2)^{1/2}}, & 0<t<\gamma,\\[2ex] \dfrac{t^3(t^2-\gamma^2)(1-t^2)^{1/2}}{(t^2-\frac{1}{2})^4+t^4(t^2-\gamma^2)(1-t^2)}, & \gamma<t<1,\end{cases}$$

$$\gamma = k_1/k_2, \qquad s_R = k_2 s. \tag{29}$$

We note that $F_n(n = 0, 1)$, defined in (29), is similar to that occurring in [3] and [7]. Also, $F_0 = \pi$. Then using the values of c_0 from (17) and F_0 we get

$$I_1 = ik_2c_0ab\pi F_1 + o(k_2^2). \tag{30}$$

Then (21) and (30) gives, similar to (14),

$$g_1(x,y) = \frac{ic_0a\pi}{2\beta K}\frac{H(1-x^2/a^2-y^2/b^2)}{(1-x^2/a^2-y^2/b^2)^{1/2}}. \tag{31}$$

A similar analysis gives

$$I_2 = \frac{c_0}{2}bk_2^2K(x^2+y^2) + c_0a^2bF_2Ek_2^2 - \frac{c_0}{2K}a^2b\pi F_1^2k_2^2$$
$$+\frac{c_0}{2}\frac{b}{k_0^2}k_2^2(x^2-y^2)[2E-(1+k_0'^2)K] + o(k_2^3), \tag{32}$$

$k_0' = b/a$. Thus

$$\iint_S \frac{g_2(x',y')}{R} \mathrm{d}s = \frac{I_2}{\omega^2}. \tag{33}$$

As in [8] we seek a solution of $g_2(x, y)$ in the form

$$g_2(x,y) = \frac{c_{20} + c_{21}x^2 + c_{22}y^2}{(1 - x^2/a^2 - y^2/b^2)^{1/2}} H\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{34}$$

Constants c_{20}, c_{21} and c_{22} can be determined on equating coefficients and written down from [8]. However, for our problem we are interested only in $\iint_S g_2(x', y')\,\mathrm{d}S$. For this we need not compute $g_2(x, y)$ directly and can proceed as follows.

Multiplying (33) by $\iint_S g_0(x, y)\,\mathrm{d}S$ and interchanging as in [9], we get

$$\iint_S g_2(x',y')\,\mathrm{d}S \iint_S \frac{g_0(x,y)}{R} \mathrm{d}S = \frac{1}{\omega^2} \iint_S g_0(x,y) I_2 \,\mathrm{d}S.$$

Using (14), we get

$$-\frac{w_0}{2(1-\nu)} \iint_S g_2(x',y')\,\mathrm{d}S = \frac{1}{\omega^2} \iint_S I_2 g_0(x,y)\,\mathrm{d}S. \tag{35}$$

On carrying out the requisite integrations which are elementary and using the value of $g_0(x, y)$ from (16) we get

$$\iint_S g_2(x',y')\,\mathrm{d}S = \frac{\pi a^3 c_0}{2\beta^2 K}\left[\frac{4}{3}F_2 E - \frac{F_1^2 \pi}{2K}\right]. \tag{36}$$

The total load P to be applied on the disc is given by

$$P = -\iint_S \tau_{zz}(x,y,0)\,\mathrm{d}x\,\mathrm{d}y. \tag{37}$$

From (6), (8) and (9) one can write

$$\begin{aligned} P &= -2\mu \iint_S \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi,\eta)\mathrm{e}^{-i(\xi x+\eta y)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x\,\mathrm{d}y \\ &= -4\pi\mu \iint_S g(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2\mu w_0 \pi a}{(1-\nu)K}\left[1 + \frac{ik_2 a F_1}{2K} + \frac{k_2^2 a^2}{2\pi}\left(\frac{4}{3}F_2 E - \frac{F_1^2\pi}{2K}\right) + o(k_2^3)\right]. \end{aligned} \tag{38}$$

The reciprocal of the total load is given by

$$\frac{1}{P} = \frac{(1-\nu)K}{2\mu w_0 \pi a}\left[1 - \frac{ik_2 a F_1}{2K} - \frac{2k_2^2 a^2}{3\pi K} F_2 E + o(k_2^3)\right]. \tag{39}$$

This agrees with the corresponding expression given in [7] [eqn (39)], up to $o(k_2^3)$ if we interchange b with a and replace ω by $-\omega$ to conform to the notation of Stallybrass and Scherer[7].

5. HORIZONTAL MOTION OF THE DISC

We now consider the more complicated case of translational motion in the x-direction. The boundary conditions, at $z = 0$, are

$$\tau_{zz} = 0, \qquad \text{for all } x, y, \tag{40a}$$

$$\tau_{zx} = \tau_{zy} = 0, \qquad (x, y) \notin S, \tag{40b}$$

$$u_x = u_0 = \text{constant}, \qquad (x\ y) \in S, \tag{40c}$$

$$u_y = 0, \qquad (x, y) \in S.$$

(40a) gives, from (6),

$$(2\xi^2 + 2\eta^2 - k_2^2)A_1(\xi, \eta) = 2\nu_2(\xi^2 + \eta^2)B_1(\xi, \eta). \tag{41}$$

Setting (41) in the boundary conditions (40b) and (40c) and defining new unknowns $B(\xi, \eta)$, $C(\xi, \eta)$ such that

$$B(\xi, \eta) = \frac{4B_1(\xi, \eta)F(\xi, \eta)}{(2\xi^2 + 2\eta^2 - k_2^2)}, \qquad C(\xi, \eta) = C_1(\xi, \eta)\,\nu_2, \tag{42}$$

we get the following system of dual integral equations:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [-i\xi B(\xi, \eta) + i\eta C(\xi, \eta)]e^{-i(\xi x + \eta y)}\, d\xi\, d\eta = 0, \qquad (x, y) \notin S, \tag{43a}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [i\eta B(\xi, \eta) + i\xi C(\xi, \eta)]e^{-i(\xi x + \eta y)}\, d\xi\, d\eta = 0, \qquad (x, y) \notin S, \tag{43b}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-\frac{i\xi\nu_2 k_2^2}{4F(\xi,\eta)}B(\xi,\eta) - \frac{i\eta}{\nu_2}C(\xi,\eta)\right]e^{-i(\xi x + \eta y)}d\xi d\eta = u_0, \qquad (x,y)\in S, \tag{43c}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{-i\eta\nu_2 k_2^2}{4F(\xi,\eta)}B(\xi,\eta) + \frac{i\xi}{\nu_2}C(\xi,\eta)\right]e^{-i(\xi x + \eta y)}d\xi d\eta = 0, \qquad (x,y)\in S. \tag{43d}$$

In order to solve the above simultaneous system of dual integral equations we consider first the solution of corresponding dual integral equations for static loading. Thus let the static solution be represented by

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [-i\xi B_0(\xi, \eta) + i\eta C_0(\xi, \eta)]e^{-i(\xi x + \eta y)}\, d\xi\, d\eta &= f_0(x, y), \\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [i\eta B_0(\xi, \eta) + i\xi C_0(\xi, \eta)]e^{-i(\xi x + \eta y)}\, d\xi\, d\eta &= h_0(x, y), \end{aligned} \tag{44}$$

where $f_0(x, y) = h_1(x, y) = 0$, $(x, y) \notin S$. Inverting (44) we get

$$\begin{aligned} iB_0(\xi, \eta) &= \frac{-\xi\overline{F}_0(\xi, \eta) + \eta\overline{H}_0(\xi, \eta)}{\xi^2 + \eta^2}, \\ iC_0(\xi, \eta) &= \frac{\eta\overline{F}_0(\xi, \eta) + \xi\overline{H}_0(\xi, \eta)}{\xi^2 + \eta^2}, \end{aligned} \tag{45}$$

where

$$\bar{F}_0(\xi, \eta) = \frac{1}{2\pi} \iint_S f_0(x, y) e^{i(\xi x' + \eta y')} \, dx' \, dy',$$
$$\bar{H}_0(\xi, \eta) = \frac{1}{2\pi} \iint_S h_0(x', y') e^{i(\xi x' + \eta y')} \, dx' \, dy'. \tag{46}$$

Substituting (44) in the limiting form of (43c) and (43d) we get

$$- \left[(1 - \nu) \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \iint_S f_0(x', y') R \, dS$$
$$- \nu \frac{\partial^2}{\partial x \, \partial y} \iint_S h_0(x', y') R \, dS = u_0,$$
$$\nu \frac{\partial^2}{\partial x \, \partial y} \iint_S f_0(x', y') R \, dS$$
$$+ \left[(1 - \nu) \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right] \iint_S h_0(x', y') R \, dS = 0. \tag{47}$$

Solutions of $f_0(x, y)$ and $h_0(x, y)$ are obtained as

$$f_0(x, y) = \frac{c_1 H(1 - x^2/a^2 - y^2/b^2)}{(1 - x^2/a^2 - y^2/b^2)^{1/2}},$$
$$h_0(x, y) = 0, \tag{48}$$

where

$$c_1 = - \frac{u_0}{[(1 - \nu)(\partial^2/\partial x^2) + \partial^2/\partial y^2] J}. \tag{49}$$

Substituting the value of

$$J = \iint_S \frac{R \, dS}{(1 - x'^2/a^2 - y'^2/b^2)^{1/2}}$$

listed in the Appendix, we get, after some simplication,

$$c_1 = - \frac{u_0 k_0^2}{2\pi b[(1 - \nu)(E - k_0'^2 K) + K - E]}. \tag{50}$$

In order to get the solution of the system of dual integral equations, we assume

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [-i\xi B(\xi, \eta) + i\eta C(\xi, \eta)] e^{-i(\xi x + \eta y)} \, d\xi \, d\eta = f(x, y),$$
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [i\eta B(\xi, \eta) + i\xi C(\xi, \eta)] e^{-i(\xi x + \eta y)} \, d\xi \, d\eta = h(x, y), \tag{51}$$

where

$$f(x, y) = h(x, y) = 0, \qquad (x, y) \notin S.$$

Then (43a) and (43b) are automatically satisfied. Substituting (51) in (43c) and (43d), after separating out the statical part, we get on rearranging that

$$
\begin{aligned}
&-\left[(1-\nu)\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right]\iint_S f(x',y')R\,\mathrm{d}S\\
&-\nu\frac{\partial^2}{\partial x\,\partial y}\iint_S h(x',y')R\,\mathrm{d}S=\epsilon_1(x,y),\\
&-\nu\frac{\partial^2}{\partial x\,\partial y}\iint_S f(x',y')R\,\mathrm{d}S\\
&-\left[(1-\nu)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\iint_S h(x',y')R\,\mathrm{d}S=\epsilon_2(x,y),
\end{aligned}
\tag{52}
$$

where

$$
\begin{aligned}
\epsilon_1(x,y)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\nu_2}{4F(\xi,\eta)}-\frac{(1-\nu)}{2(\xi^2+\eta^2)^{1/2}}\right]\\
&\times\frac{k_2^2}{(\xi^2+\eta^2)}\,\mathrm{e}^{-i(\xi x+\eta y)}\left[\xi^2\iint_S f(x',y')\mathrm{e}^{i(\xi x'+\eta y')}\,\mathrm{d}S\right.\\
&\left.-\xi\eta\iint_S h(x',y')\mathrm{e}^{i(\xi x'+\eta y')}\,\mathrm{d}S\right]\mathrm{d}\xi\,\mathrm{d}\eta\\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{\nu_2}-\frac{1}{(\xi^2+\eta^2)^2)^{1/2}}\right]\frac{\mathrm{e}^{-i(\xi x+\eta y)}}{(\xi^2+\eta^2)}\\
&\times\left[\eta^2\iint_S f(x',y')\mathrm{e}^{i(\xi x'+\eta y')}\,\mathrm{d}S\right.\\
&\left.+\xi\eta\iint_S h(x',y')\mathrm{e}^{i(\xi x'+\eta y')}\,\mathrm{d}S\right]\mathrm{d}\xi\,\mathrm{d}\eta,\\
\epsilon_2(x,y)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\nu_2}{4F(\xi,\eta)}-\frac{1-\nu}{2(\xi^2+\eta^2)^{1/2}}\right]\\
&\times\frac{k_2^2}{(\xi^2+\eta^2)}\,\mathrm{e}^{-i(\xi x+\eta y)}\left[-\xi\eta\iint_S f(x',y')\mathrm{e}^{i(\xi x'+\eta y')}\,\mathrm{d}S\right.\\
&\left.+\eta^2\iint_S h(x',y')\mathrm{e}^{i(\xi x'+\eta y')}\,\mathrm{d}S\right]\mathrm{d}\xi\,\mathrm{d}\eta\\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{\nu_2}-\frac{1}{(\xi^2+\eta^2)^{1/2}}\right]\frac{\mathrm{e}^{-i(\xi x+\eta y)}}{(\xi^2+\eta^2)}\\
&\times\left[\xi\eta\iint_S f(x',y')\mathrm{e}^{i(\xi x'+\eta y')}\,\mathrm{d}S\right.\\
&\left.+\xi^2\iint_S h(x',y')\mathrm{e}^{i(\xi x'+\eta y')}\,\mathrm{d}S\right]\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned}
\tag{53}
$$

A low-frequency solution of (52) will now be presented. To this end assume that $f(x, y)$ and $h(x, y)$ have the following low-frequency expansions, for $k_2 a \ll 1$:

$$
\begin{aligned}
f(x,y)&=f_0(x,y)+\omega f_1(x,y)+\omega^2 f_2(x,y)+\cdots,\\
h(x,y)&=h_0(x,y)+\omega h_1(x,y)+\omega^2 h_2(x,y)+\cdots,
\end{aligned}
\tag{54}
$$

where $f_0(x, y)$ and $h_0(x, y)$ are given by (48). Substituting (54) in (52) and noting that the integrands in each of the integrals in (53), by virtue of construction, have a first term of order ω in the low-frequency expansion, we get the following integrodifferential equations for determining $f_1(x, y)$ and $h_1(x, y)$ upon equating coefficients of ω on both sides of (52):

$$\begin{aligned}
&-\left[(1-\nu)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\iint_S f_1(x',y')R\,\mathrm{d}S \\
&-\nu\frac{\partial^2}{\partial x\,\partial y}\iint_S h_1(x',y')R\,\mathrm{d}S = P_1 \\
&-\nu\frac{\partial^2}{\partial x\,\partial y}\iint_S f_1(x',y')R\,\mathrm{d}S \\
&-\left[(1-\nu)\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right]\iint_S h_1(x',y')R\,\mathrm{d}S = Q_1.
\end{aligned} \tag{55}$$

In (55) ωP_1 and ωQ are the first terms in the low-frequency expansions of $\epsilon_1^0(x, y)$ and $\epsilon_2^0(x, y)$, which are obtained from $\epsilon_1(x, y)$ and $\epsilon_2(x, y)$ in (53) on replacing $f(x, y)$ and $h(x, y)$ by their zero-order solution, namely, $f_0(x, y)$ and $h_0(x, y)$, respectively.

In a similar fashion, any higher-order $f_n(x, y)$ and $h_n(x, y)$ are related to lower orders $f_{n-1}(x, y)$ and $h_{n-1}(x, y)$ through a similar integrodifferential equation like (55).

Before proceeding to get a solution of (55) it is necessary to find P_1 and Q_1. To do this we make the substitution

$$\xi = k\cos\chi, \qquad \eta = k\sin\chi,$$
$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Then

$$\epsilon_1^0(x, y) = u_0 - J_{11} - J_{12}, \tag{56}$$

where

$$\begin{aligned}
J_{11} &= c_1 ab\left(\frac{\pi}{2}\right)^{1/2}\int_0^{2\pi}\int_0^{\infty}\left[\frac{\nu_2 k_2^2\cos^2\chi}{4F(k)}-\frac{\sin^2\chi}{\nu_2}\right]J_0(kr) \\
&\quad\times\frac{\sin[k(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}]}{(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}\,\mathrm{d}k\,\mathrm{d}\chi, \\
J_{12} &= c_1 ab\left(\frac{\pi}{2}\right)^{1/2}\sum_{n=1}^{\infty}\int_0^{2\pi}\int_0^{\infty}\left[\frac{\nu_2 k_2^2\cos^2\chi}{4F(k)}-\frac{\sin^2\chi}{\nu_2}\right] \\
&\quad\times J_{2n}(kr)\frac{\sin[k(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}]}{(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}\,\mathrm{d}k\,\mathrm{d}\chi.
\end{aligned} \tag{57}$$

Following a method similar to that in Section 4, we can show that

$$\begin{aligned}
J_{11} = &-\frac{c_1 ab}{4}\int_0^{2\pi}\left[\int_0^{k_1}\frac{k_2^2(k_2^2-k^2)^{1/2}J_0(kr)\mathrm{e}^{-ik(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}}{(k^2-k_2^2/2)^2+k^2(k_1^2-k^2)^{1/2}(k_2^2-k^2)^{1/2}}\mathrm{d}k\right. \\
&+\int_{k_1}^{k_2}\frac{k_2^2(k^2-k_2^2/2)^2(k_2^2-k^2)^{1/2}J_0(kr)\mathrm{e}^{-ik(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}}{(k^2-k_2^2/2)^4+k^4(k_2^2-k^2)(k_1^2-k^2)}\mathrm{d}k \\
&\left.-\pi\frac{(s_R^2-k_2^2)^{1/2}k_2^2}{F'(s_R)}\mathrm{e}^{-is_R(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}\right]\frac{\cos^2\chi\,\mathrm{d}\chi}{(a^2\cos^2\chi+b^2\sin\chi)^{1/2}} \\
&-c_1 ab\int_0^{2\pi}\int_0^{k_2}\frac{J_0(kr)}{(k_2^2-k^2)^{1/2}}\mathrm{e}^{-ik(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}\frac{\sin^2\chi}{(a^2\cos^2\chi+b^2\sin^2\chi)^{1/2}}\mathrm{d}k\,\mathrm{d}\chi,
\end{aligned} \tag{58}$$

which, when expanded in powers of k_2, gives

$$J_{11} = abc_1 \left[- \frac{A_0(1-\nu)}{2} \int_0^{2\pi} \frac{\cos^2\chi \, d\chi}{(a^2\cos^2\chi + b^2\sin^2\chi)^{1/2}} + \frac{ik_2A_1(1-\nu)\pi}{2} - \frac{\pi}{2}\int_0^{2\pi} \frac{\sin^2\chi \, d\chi}{(a^2\cos^2\chi + b^2\sin^2\chi)^{1/2}} + ik_2\pi + o(k_2^{\frac{3}{2}}) \right], \tag{59}$$

where, for $n = 0, 1, 2$,

$$A_n = \frac{1}{2(1-\nu)} \left[- \frac{\pi s^n(s^2-1)^{1/2}}{F'(s)} + \int_0^{\gamma} \frac{x^n(1-x^2)^{1/2}\,dx}{(x^2-\frac{1}{2})^2 + x^2(\gamma^2-x^2)^{1/2}(1-x^2)^{1/2}} + \int_{\gamma}^{1} \frac{x^2(x^2-\frac{1}{2})^2(1-x^2)^{1/2}\,dx}{(x^2-\frac{1}{2})^4 + x^4(x^2-\gamma^2)(1-x^2)}, \right. \tag{60}$$

$\gamma = k_2/k_1$, $s_R = k_2 s$. We note that A_n $(n = 0, 1)$ as given by (60) is identical with similar quantity in [3].

Noting that $A_0 = \pi$ and using tables of elliptic functions, we get

$$\begin{aligned} J_{11} &= - \frac{2\pi c_1 ab}{k_0^2} [(1-\nu)(E - k_0'^2 K) + (K - E)] \\ &\quad + \frac{ik_2}{2}\pi c_1 ab\,[A_1(1-\nu) + 2] + o(k_2^{\frac{3}{2}}) \\ &= u_0 + \frac{ik_2}{2}\pi[A_1(1-\nu) + 2]c_1 ab + o(k_2^{\frac{3}{2}}), \end{aligned} \tag{61}$$

on using (50). Similarly it can be shown that

$$J_{12} \sim o(k_2^{\frac{3}{2}}). \tag{62}$$

Also, $\epsilon_2^0(x, y) \sim o(k_2^{\frac{3}{2}})$.

From (56), (61) and (62) we obtain

$$P_1 = - \frac{i}{2\beta}[A_1(1-\nu) + 2]c_1 ab\pi, \qquad Q_1 = 0. \tag{63}$$

Then (55) gives, similar to (47),

$$\begin{aligned} f_1(x, y) &= \frac{i\pi c_1 ab[A_1(1-\nu) + 2]}{2\beta\,[(1-\nu)(\partial^2/\partial x^2) + \partial^2/\partial y^2]J} \times \frac{H(1 - x^2/a^2 - y^2/b^2)}{(1 - x^2/a^2 - y^2/b^2)^{1/2}}, \\ h_1(x, y) &= 0. \end{aligned} \tag{64}$$

We could easily obtain higher-order approximations. However, since the computations are rather involved, we restrict ourselves to first-order results.

The quantity of physical interest is the load on the disc, which is given by

$$P = - \tau_{zx}(x, y, 0) = - \mu \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [-i\xi B(\xi, \eta) + i\eta C(\xi, \eta)] e^{-i(\xi x + \eta y)}\, d\xi\, d\eta = - 2\mu\pi f(x, y). \tag{65}$$

Then from (54), (48) and (64) we get

$$P = \frac{\mu u_0 k_0^2}{b[(1-\nu)(E - k_0'^2 K) + K - E]} \times \left\{1 + \frac{ik_2 a k_0^2[A_1(1-\nu)+2]}{4[(1-\nu)(E-k_0'^2K)+K-E]}\right\} \frac{H(1 - x^2/a^2 - y^2/b^2)}{(1 - x^2/a^2 - y^2/b^2)^{1/2}}. \quad (66)$$

Thus the total load, which is a constant multiple of the dynamic compliance, is given by

$$\iint_S P\, dx\, dy = \frac{2\pi\mu u_0 k_0^2 a}{(1-\nu)(E-k_0'^2K)+K-E} \times \left\{1 + \frac{ik_2 k_0^2 a[A_1(1-\nu)+2)]}{4[(1-\nu)(E-k_0'^2K)+K-E]}\right\}. \quad (67)$$

6. DISCUSSION

In the limiting case of a circle ($b \to a$) exhibiting horizontal vibration of constant amplitude, the total load given by (67) agrees with that in [3] when due consideration is given to the fact that in our case a harmonic time dependence $e^{i\omega t}$ is assumed.

We have thus shown how the low-frequency expansions of the dynamic compliances can be obtained for horizontal and vertical vibration, once the static loading case is known. However, the evaluations of terms beyond k_2^2 in case of vertical vibration and beyond k_2 in case of horizontal vibration become rather involved. Since the expression for the load on the disc is, however, known from our work, perhaps a combination of variational techniques used in [7] and the present method can be used for obtaining terms beyond k_2^2.

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APPENDIX

The following integrals have been used in the text:

$$I = \iint_S \frac{dx'\, dy'}{R(1 - x'^2/a^2 - y'^2/b^2)^{1/2}},$$

$$J = \iint_S \frac{R\, dx'\, dy'}{(1 - x'^2/a^2 - y'^2/b^2)^{1/2}},$$

where $R = [(x - x')^2 + (y - y')^2]^{1/2}$ and S is the elliptic region $x^2/a^2 + y^2/b^2 \leq 1$.

Details of evaluation of the integral have been considered in [8]. Integrals can be evaluated after a polar coordinate system relative to (x, y) is introduced and then evaluating either directly or after reducing the integral to a branch line integral and evaluating by the residue method

In terms of elliptic functions of the first and second kinds with argument $k_0 = (1 - b^2/a^2)^{1/2}$ the values of the integral are

$$I = 2\pi b K,$$

$$J = \frac{\pi b^3 E}{k_0'^2} + \frac{\pi x^2 b}{k_0^2}(E - k_0'^2 K) + \frac{\pi y^2 b}{k_0^2}(K - E),$$

$$k_0' = (1 - k_0^2)^{1/2} = b/a$$